# Vladimir M. Miklyukov

(Volgograd State University, RUSSIA)


# ON ALMOST-SOLUTIONS OF PDE

Below we bring a series of results concerned with almost-solutions of nonlinear PDE. In most applications of PDE to the natural science, we work with "imperfect" solutions that is functions which are only "near" to true solutions. In the process of calculations, we also find only a function "near" to the true solution.

Let $D \subset \mathbb{R}^n$ be a domain and let $k(x) : D \to \mathbb{R}^1$ be a measurable function such that for an arbitrary subdomain $D' \subset\subset D$ the following property holds

$$0 < \operatorname*{ess\,inf}_{D'} k(x) \le \operatorname*{ess\,sup}_{D'} k(x) < \infty \, .$$

Let $A(x, \xi) : D \times \mathbf{R}^n \to \mathbb{R}^1$ be a measurable vector-function with properties

$$\nu_1 \, k(x) \, |\xi|^p \le \langle \xi, A(x, \xi) \rangle,$$

$$|A(x, \xi)| \le k(x) \, \nu_2 \, |\xi|^{p-1},$$

where $p \ge 1$ and $\nu_1$, $\nu_2 > 0$ are some constants.

Generalized solutions $f : D \to \mathbb{R}^1$ of the elliptic equation

$$\operatorname{div} A(x, \nabla f) = 0 \qquad (*)$$

are functions $f \in W^{1,p}_{\rm loc}(D)$ such that

$$\int\limits_D \langle \nabla \varphi, A(x, \nabla f) \rangle dx_1 \cdots dx_n = 0 \quad \forall \, \varphi \in W^{1,q}_0(D), \; \frac{1}{p} + \frac{1}{q} = 1.$$

In practical calculations instead of derivatives we work with difference quotients and instead of precise equalities with inequalities.

Almost-solutions of (*) with a deviation $\varepsilon > 0$ are defined as functions $f \in W^{1,p}_{\rm loc}(D)$ for which

$$\left| \int\limits_D \langle \nabla \varphi, A(x, \nabla f) \rangle dx_1 \cdots dx_n \right| < \varepsilon, \quad \forall \, \varphi \in W^{1,q}_0(D), \;\; |\varphi| < 1.$$

For $p > 1$ the equation (*) contains the $p$-harmonic functions equation ( see Ch, 6 in the book of J. heinonen, T. Kilpeläinen and O. Martio "Nonlinear Potential Theory of Degenerate Elliptic Equations", Clarendon Press, 1993). The assumption $p = 1$ permits to consider the minimal surface equation, the equation of maximal surfaces in Minkowski space and the gas dynamics equation.

It is clear that every $C^2$-function $h : D \to \mathbb{R}^1$ such that

$$|\operatorname{div} A(x, \nabla h)| \le \varepsilon_1,$$

is an almost-solution of (*) with the deviation $\varepsilon_1 \, \mathcal{H}^n \, (D)$.

The concept of almost-solutions was introduced in the article "$A$-Solutions with singularities as almost-solutions. Sbornik: Mathematics. v. 197. n. 11. 2006. 1587-1605"with a view to research singularities of solutions (*). We prove that some $A$-solutions with nonremovable singularities are almost-solutions.

In the article "Almost-quasiconformal mappings as almost-solutions. In Math. and Appl. Analysis. N. 3. Tyumen. State. un-t. 2007. 59-70"we connect almost-quasiconformal mappings in the E.D. Callender sense with almost-solutions of (*); in "Some conditions of differentiabilty of almost-quasiconformal maps at a point, In Notes of Seminar 'Superslow Processes', n. 4, Volgograd: Izd-vo VolGU, 2009", we research properties of such mappings.

In the article "Maximum principle for difference of almost-solutions of nonlinear elliptic equations. Vestnik Tomsk Gosud. un-ta. Math. and Mech. N 1. 2007 33-45" we prove the following special form of the maximum principle for differences of almost-solutions.

**Theorem A.** *Let $h_1\,, h_2$ be almost-solutions* (*) *with deviations $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ in a bounded domain $D \subset \mathbb{R}^n$, such that*

$$\limsup_{\substack{x\to x_0 \\ x\in D,\, x_0\in\partial D}} \ (h_1(x) - h_2(x)) \le 0 \quad \forall x_0 \in \partial D.$$

*Then either $h_1(x) \le h_2(x)$ everywhere on $D$ or the open set*

$$\mathcal{O} = \{x \in D : (h_1(x) - h_2(x)) > 0\}$$

*is not empty and*

$$\int\limits_{\{|x|<r\}\cap\mathcal{O}} k(x)\,|\nabla(h_2 - h_1)|^2\,d\mathcal{H}^n \le \frac{2M}{\mu_1}\,(\varepsilon_1 + \varepsilon_2)\,, \quad M = \sup_D |h_2(x) - h_1(x)|\,.$$

In articles "Stagnation zones and almost-solutions of elliptic equations, "Function theory, its applications and adjacent queations", Proc. of Lobachevski math. center, v. 35, Kazan: Kazan math. soc., 2007, 174-181", "On Stagnation Zones in Superslow Processes, Doklady Mathematics, v. 77, n. 1, 2008, 55-58"and "Estimates of sizes of stagnation zones of almost-solutions of elliptic and paraboliv types, Sib. j. of industr. math., v. XI, n. 3(35), 2008, 96-101", we show sizas of stagnation zones of almost-solutions.

In the article "On Harnack inequality for almost-solutions of elliptic equations, Izv. RAN, Ser. math., v. 73, n. 5, 2009"we prove the following spacial form of the Harnack's inequality.

**Theorem B.** *Let $D$ be a domain in $\mathbb{R}^n$ and let $U, V$ be its subdomains, $V \Subset U \Subset D$. Let $h$ be a positive almost-solution in $D$ of* (*) *with $k \equiv 1$, $p > n - 1$ and*

$$A(x, \lambda\xi) = \lambda\,|\lambda|^{p-2}\,A(x,\xi) \quad \forall x \in D \text{ and } \forall\,\lambda \in \mathbb{R}^1\,. \tag{**}$$

*Then*

$$\inf_{\mathcal{O}_C} \max\{h(x) : x \in V \setminus \mathcal{O}_C\} \le \exp\{\,\theta_p(V,U,D)\}\,\sup_{\mathcal{O}_C} \min\{h(x) : x \in V \setminus \mathcal{O}_C\}\,,$$

*where infimum and maximum are taken over all nonempty open subsets $\mathcal{O}_C \subset D$ such that $h|_{\partial\mathcal{O}_C} = C$, $C = \text{const}$, and $\theta_p(V,U,D)$ is a constant* (*the form of which is showed*).

In the article "Solutions of parabolic equations as almost-solutions of elliptic, In Math. and Appl. Analysis, Tyumen. State. un-t. 2009, to app."we connect solutions of parabolic equations with almost-solutions of elliptic equations.

**Theorem C.** *Let* $h = h(x,t) : D \times (\tau_0, \tau_1) \to \mathbb{R}^1$ *be a generalized solution of*

$$\operatorname{div} A(x, \nabla h) = B(t, h, h'_t)\,,$$

*where* $A(x, \xi)$ *satisfies to* (**),

$$B(t, h, h'_t) = b_0(t)\,|h|^{p-2}\,h + b_1(t)\,|h|^{p-2}\,h\frac{\partial h}{\partial t}(x,t)$$

*and*

$$b_0(t) > 0,\; b_1(t) : (\tau_0, \tau_1) \subset \mathbb{R}^1 \to \mathbb{R}^1$$

*be locally Lipschitz on* $(\tau_0, \tau_1)$ *functions.*

*Then* $h(x,t)$ *is an almost-solution of an equation* (*), *and the deviationn* $s(\tau_0, \tau_1)$ *of tha almost-solution is*

$$s(\tau_0, \tau_1) = \int\limits_D d\mathcal{H}^n \int\limits_{\tau_0}^{\tau_1} \left| b_0\,|h|^{p-2}h + b_1\,|h|^{p-2}h\,h'_t - \frac{d}{dt}\left(b_1\,|h'_t|^{p-2}h'_t\right)\right| \,dt\,.$$

Near statements are for solutions of hyperbolic equations.

Some applications to questions of singularities removability for gas dinamics equations and quasiregular mappings, see in Chapt. 7 of our book "Geometric Analysis: Differential Forms, Almost-Solutions and Almost-Quasiconformal Maps. Volgograd: Izd-vo VolGU, 2007".

Scientific head of the laboratory
"Superslow processes"
Prof. Vladimir M. Miklyukov
miklyuk@mail.ru
June 07 2009