ON ALMOST SOLUTIONS OF PDE

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Below we bring a series of results concerned with almost solutions of nonlinear PDE. In most applications of PDE to the natural science, we work with "imperfect" solutions that is functions which are only "near" to true solutions. In the process of calculations, we also find only a function "near" to the true solution.

Let $D \subset \mathbb{R}^n$ be a domain and let $k(x) : D \to \mathbb{R}^1$ be a measurable function such that for an arbitrary subdomain $D' \subset\subset D$ the following property holds

$$0 < \operatorname{ess\,inf}_{D'} k(x) \leq \operatorname{ess\,sup}_{D'} k(x) < \infty .$$

Let $A(x, \xi) : D \times \mathbf{R}^n \to \mathbb{R}^1$ be a measurable vector-function with properties

$$\nu_1\, k(x)\, |\xi|^p \leq \langle \xi, A(x, \xi) \rangle,$$

$$|A(x, \xi)| \leq k(x)\, \nu_2\, |\xi|^{p-1},$$

where $p \geq 1$ and ν_1, $\nu_2 > 0$ are some constants.

Generalized solutions $f : D \to \mathbb{R}^1$ of the elliptic equation

$$\operatorname{div} A(x, \nabla f) = 0 \qquad (*)$$

are functions $f \in W^{1,p}_{\text{loc}}(D)$ such that

$$\int_D \langle \nabla \varphi, A(x, \nabla f) \rangle dx_1 \cdots dx_n = 0 \quad \forall\, \varphi \in W^{1,q}_0(D),\ \frac{1}{p} + \frac{1}{q} = 1.$$

In practical calculations instead of derivatives we work with difference quotients and instead of precise equalities with inequalities.

Almost solutions of (*) with a deviation $\varepsilon > 0$ are defined as functions $f \in W^{1,p}_{\text{loc}}(D)$ for which

$$\left| \int_D \langle \nabla \varphi, A(x, \nabla f) \rangle dx_1 \cdots dx_n \right| < \varepsilon, \quad \forall\, \varphi \in W^{1,q}_0(D),\ \ |\varphi| < 1.$$

For $p > 1$ the equation (*) contains the p-harmonic functions equation (see Ch, 6 in the book of J. heinonen, T. Kilpeläinen and O. Martio "Nonlinear Potential Theory of Degenerate Elliptic Equations", Clarendon Press, 1993). The assumption $p = 1$ permits to consider the minimal surface equation, the equation of maximal surfaces in Minkowski space and the gas dynamics equation.

It is clear that every C^2-function $h : D \to \mathbb{R}^1$ such that

$$|\operatorname{div} A(x, \nabla h)| \leq \varepsilon_1,$$

is an almost solution of (*) with the deviation $\varepsilon_1\, \mathcal{H}^n\,(D)$.

The concept of almost solutions was introduced in the article "A-Solutions with singularities as almost solutions. Sbornik: Mathematics. v. 197. n. 11. 2006. 1587-1605"with a view to research singularities of solutions (*). We prove that some A-solutions with nonremovable singularities are almost solutions.

In the article "Almost quasiconformal mappings as almost solutions. In Math. and Appl. Analysis. N. 3. Tyumen. State. un-t. 2007. 59-70"we connect almost quasiconformal mappings in the E.D. Callender sense with almost solutions of (*); in "Some conditions of differentiabilty of almost quasiconformal maps at a point, In Notes of Seminar 'Superslow Processes', n. 4, Volgograd: Izd-vo VolGU, 2009", we research properties of such mappings.

In the article "Maximum principle for difference of almost solutions of nonlinear elliptic equations. Vestnik Tomsk Gosud. un-ta. Math. and Mech. N 1. 2007 33-45" we prove the following special form of the maximum principle for differences of almost solutions.

Theorem A. *Let h_1, h_2 be almost solutions (*) with deviations $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ in a bounded domain $D \subset \mathbb{R}^n$, such that*

$$\limsup_{\substack{x \to x_0 \\ x \in D,\, x_0 \in \partial D}} (h_1(x) - h_2(x)) \le 0 \quad \forall x_0 \in \partial D.$$

Then either $h_1(x) \le h_2(x)$ everywhere on D or the open set

$$\mathcal{O} = \{x \in D : (h_1(x) - h_2(x)) > 0\}$$

is not empty and

$$\int\limits_{\{|x|<r\} \cap \mathcal{O}} k(x)\, |\nabla (h_2 - h_1)|^2 \, d\mathcal{H}^n \le \frac{2M}{\nu_1} (\varepsilon_1 + \varepsilon_2), \quad M = \sup_D |h_2(x) - h_1(x)| .$$

In articles "Stagnation zones and almost solutions of elliptic equations, "Function theory, its applications and adjacent queations", Proc. of Lobachevski math. center, v. 35, Kazan: Kazan math. soc., 2007, 174-181", "On Stagnation Zones in Superslow Processes, Doklady Mathematics, v. 77, n. 1, 2008, 55-58"and "Estimates of sizes of stagnation zones of almost solutions of elliptic and paraboliv types, Sib. j. of industr. math., v. XI, n. 3(35), 2008, 96-101", we show sizas of stagnation zones of almost solutions.

In the article "On Harnack inequality for almost solutions of elliptic equations, Izv. RAN, Ser. math., v. 73, n. 5, 2009"we prove the following spacial form of the Harnack's inequality.

Theorem B. *Let D be a domain in $\mathbb{R}^n$ and let U, V be its subdomains, $V \Subset U \Subset D$. Let h be a positive almost solution in D of (*) with $k \equiv 1$, $p > n-1$ and*

$$A(x, \lambda\xi) = \lambda\, |\lambda|^{p-2} A(x, \xi) \quad \forall x \in D \text{ and } \forall\, \lambda \in \mathbb{R}^1 . \tag{**}$$

Then

$$\inf_{\mathcal{O}_C} \max\{h(x) : x \in V \setminus \mathcal{O}_C\} \le \exp\{\,\theta_p(V, U, D)\} \sup_{\mathcal{O}_C} \min\{h(x) : x \in V \setminus \mathcal{O}_C\} ,$$

where infimum and maximum are taken over all nonempty open subsets $\mathcal{O}_C \subset D$, $D \setminus \mathcal{O}_C \ne \emptyset$, such that $h|_{\partial \mathcal{O}_C} = C$, $C = \mathrm{const}$, and $\theta_p(V, U, D)$ is a constant (the form of which is showed).

In the article "Solutions of parabolic equations as almost solutions of elliptic, In Math. and Appl. Analysis, Tyumen. State. un-t. 2009, to app."we connect solutions of parabolic equations with almost solutions of elliptic equations.

Theorem C. *Let $h = h(x,t) : D \times (\tau_0, \tau_1) \to \mathbb{R}^1$ be a generalized solution of*

$$\operatorname{div} A(x, \nabla h) = B(t, h, h'_t) ,$$

where $A(x,\xi)$ satisfies to (**),

$$B(t,h,h'_t) = b_0(t)\,|h|^{p-2}\,h + b_1(t)\,|h|^{p-2}\,h\frac{\partial h}{\partial t}(x,t)$$

and

$$b_0(t) > 0,\; b_1(t) : (\tau_0,\tau_1) \subset \mathbb{R}^1 \to \mathbb{R}^1$$

be locally Lipschitz on (τ_0,τ_1) functions.

Then $h(x,t)$ is an almost solution of an equation (*), and the deviation $s(\tau_0,\tau_1)$ of the almost solution is

$$s(\tau_0,\tau_1) = \int\limits_D d\mathcal{H}^n \int\limits_{\tau_0}^{\tau_1} \left| b_0\,|h|^{p-2}h + b_1\,|h|^{p-2}h\,h'_t - \frac{d}{dt}\left(b_1\,|h'_t|^{p-2}h'_t\right) \right| dt\,.$$

Near statements are for solutions of hyperbolic equations.

Some applications to questions of singularities removability for gas dynamics equations and quasiregular mappings, see in Chapt. 7 of our book "Geometric Analysis: Differential Forms, Almost Solutions and Almost Quasiconformal maps. Volgograd: Izd-vo VolGU, 2007".

In the paper "Three sphere theorem for almost harmonic functions"(in prep., 2010) we prove an analogue of the Hadamard's theorem on three circles for almost p-harmonic functions on domains of the ball layer type. The proof is based on the maximum principle for differences of almost p-harmonic functions.

We consider the equation

$$\operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0\,, \quad p > 1. \qquad (***)$$

The almost solutions with the deviation $\varepsilon = 0$ are called *generalized solutions*. The generalized solutions h of (***) are called also *p-harmonic* functions, and the equation (***) is called p-harmonic (see Heinonen J., Kilpeläinen T., and Martio O., *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, Oxford etc., 1993).

Let $D \subset \mathbb{R}^n$ be a domain and let $k(x) : D \to \mathbb{R}^1$ be a nonnegative measurable function.

Let A, B be nonempty closet with respect to D subsets, $A \cap B = \emptyset$. Denote by

$$\operatorname{cap}_k(A,B) = \inf_u \int\limits_D k(x)\,|\nabla u|^2\,d\mathcal{H}^n, \quad u \in C^1(D)\,, \quad u|_A \equiv 0, \quad u|_B \equiv 1\,,$$

the k-capacity of the condenser $(A,B;D)$ and by

$$\lambda_k(\mathcal{O}) = \inf_u \frac{\int\limits_{\mathcal{O}} k(x)\,|\nabla u|^2 d\mathcal{H}^n}{\int\limits_{\mathcal{O}} k(x)\,u^2 d\mathcal{H}^n}\,, \quad u \in C^1(\mathcal{O}) \cap C^0(\overline{\mathcal{O}}), \quad u|_{\partial\mathcal{O}} = 0,$$

the main k-frequency of an open set $\mathcal{O} \subset \mathbb{R}^n$.

We shall say that an unbounded domain $D \subset \mathbb{R}^n$ is *k-narrow* close to the infinity of $\mathbb{R}^n$, if for an arbitrary $r > 0$ the following property holds

$$\lim_{R\to\infty} \operatorname{cap}_k(D_r, D \setminus D_R) = 0\,,$$

where $D_t = \{|x| < t\} \cap D$.

Below we bring an generalization of the three circles theorem for p-harmonic functions $v\colon \Omega \to \mathbb{R}^1$ on j-balls in $\mathbb{R}^n$, which are defined in the following way. Fix an integer number j, $1 \le j \le n$ and a real number $t \ge 0$. The sets

$$B_j(t) = \{x \in \mathbb{R}^n : d_j(x) < t\} \text{ and } \Sigma_j(t) = \partial B_j(t), \text{ where } d_j(x) = \Big(\sum_{i=1}^{j} x_i^2\Big)^{1/2},$$

we shall call by the j-ball and the j-sphere in $\mathbb{R}^n$. For $j = n$ the ball $B_j(t)$ is the standard Euclidean ball $B^n(0,t)$ and the sphere $\Sigma_j(t)$ is the Euclidean sphere $S^{n-1}(0,t)$. In particular, the symbol $\Sigma_j(0)$ means the j-sphere of the radius 0, that is

$$\Sigma_j(0) = \{x = (x_1, \ldots, x_j, \ldots, x_n) : x_1 = \ldots = x_j = 0\}.$$

Let $0 < \alpha < \beta < \infty$ be fixed numbers and let

$$D^j_{\alpha,\beta} = \{x \in \mathbb{R}^n : \alpha < d_j(x) < \beta\}.$$

For $j = 1$ the set $D^j_{\alpha,\beta}$ is the layer between two parallel hyperplanes. For $1 < j < n$ the boundary of $D^j_{\alpha,\beta}$ consists of two cylindrical surfaces.

Let $v \in C^0(D^j_{r,R})$ and let

$$M(r) = \limsup_{x \to \Sigma_j(r)} v(x).$$

Consider the function

$$v_{r,R}(x) = \frac{v(x) - M(r)}{M(R) - M(r)}, \quad r < R.$$

Theorem D. *Let $1 < p < \infty$, $0 < r < R \le \infty$. Let $v(x) \in \mathrm{Lip}_{\mathrm{loc}}(D^j_{r,R})$ be a bounded almost solution of (***) on $D^j_{r,R}$, $1 \le j \le n$, with a deviation $\varepsilon > 0$ and let $M(t) = \sup_{\Sigma_i(t)} v(x)$. Then for all $t \in (r,R)$ such that*

$$\Sigma_j(t) \cap \mathcal{O} = \emptyset, \quad \mathcal{O} = \{x \in D^j_{r,R} : v_{r,R}(x) - u(x) > 0\},$$

the following property holds

$$M(t) \le \big(M(R) - M(r)\big) u_0^{j,p}(t) + M(r),$$

Moreover, if the oper set $\mathcal{O}$ is nonempty, then

$$\frac{1}{2} \int\limits_{\{|x|<r\}\cap\mathcal{O}} k(x)\, |\nabla(v_{r,R}(x) - u(x))|^2 \, d\mathcal{H}^n \le \frac{A}{\mu_1}\, \varepsilon +$$

$$+2 \left(\frac{\mu_2}{\mu_1}\right)^2 A^2 \operatorname{cap}_k(\mathcal{O}_r, \mathcal{O} \setminus \mathcal{O}_R),$$

where

$$A = \sup_{D^j_{r,R}} |v_{r,Q}(x) - u(x)|$$

and

$$k(x) = \int\limits_0^1 |\lambda \nabla v(x) + (1-\lambda)(M(R) - M(r))\, \nabla u(x)|^{p-2} \, d\lambda.$$

In the case $j = p = n$ we have

$$\xi(r,t) = \ln\frac{t}{r} \quad \text{и} \quad u_0^{n,n}(t) = \frac{\ln(t/r)}{\ln(R/r)}$$

and consequently Theorem D implies

Corollary 1. *Let $0 < r < R \le \infty$ and let $v(x) \in \mathrm{Lip}_{\mathrm{loc}}(D^n_{r,R})$ be a positive almost solution of (***) with $p = n$ and a deviation $\varepsilon > 0$ on a ball layer*

$$D^n_{r,R} = \{r < |x| < R\}\,.$$

Then for an arbitrary $t \in (r, R)$,

$$\Sigma_n(t) \cap \mathcal{O} = \emptyset\,, \quad \mathcal{O} = \{x \in D^n_{r,R} : v_{r,R}(x) - u(x) > 0\}\,,$$

we have

$$M(t)^{\ln(R/r)} \le M(r)^{\ln(R/t)}\, M(R)^{\ln(t/r)}\,,$$

Moreover, if the open set $\mathcal{O}$ is nonempty, then there is valid the size estimate of Theorem D.

In the paper "Two spheres theorem for almost solutions of maximal type surfaces equation"(in prep., 2010) we consider almost solutions of strong nonlinear elliptic equations. We bring a special version of the Hadamard's three circles theorem. In the different from the case of harmonic functions here for an estimate of the function maximum on an inner circle, it is enough to known the maximum only on an exterior circle, that is the corollary of the strong nonlinearityof the equation (see, for example, the chapter VI of the J.C.C. Nitsche's monograph "Vorlesungen über Minimalflächen", Springer-Verlag, Berlin Heidelberg New York, 1975).

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